

Tuning across Universalities with a Driven Open Condensate

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Driven-dissipative systems in two dimensions can differ substantially from their equilibrium counterparts. In particular, a dramatic loss of off-diagonal algebraic order and superfluidity has been predicted to occur due to the interplay between coherent dynamics and external drive and dissipation in the thermodynamic limit. We show here that the order adopted by the system can be substantially altered by a simple, experimentally viable, tuning of the driving process. More precisely, by considering the long-wavelength phase dynamics of a polariton quantum fluid in the optical parametric oscillator regime, we demonstrate that simply changing the strength of the pumping mechanism in an appropriate parameter range can substantially alter the level of effective spatial anisotropy induced by the driving laser, and move the system into distinct scaling regimes. These include: (i) the classic algebraically ordered superfluid below the Berezinskii-Kosterlitz-Thouless (BKT) transition, as in equilibrium; (ii) the non-equilibrium, long-wave-length fluctuation dominated Kardar-Parisi-Zhang (KPZ) phase; and the two associated topological defect dominated disordered phases caused by proliferation of (iii) entropic BKT vortex-antivortex pairs or (iv) repelling vortices in the KPZ phase. Further, by analysing the renormalization group flow in a finite system, we examine the length scales associated with these phases, and assess their observability in current experimental conditions.

I. INTRODUCTION

The concept of universality permits to order and classify a great variety of different physical systems in terms of their common collective behaviour in the long-wavelength limit. While dynamical critical phenomena in equilibrium are by now quite well understood [1], the extension to non-equilibrium is a relatively new field. At the same time, due to an unprecedented experimental progress on a range of light-matter realisations [2] in recent years, there is a particular interest in collective behaviour of driven-dissipative quantum systems. Despite the fact that energy is not conserved, the detailed balance condition is broken and fluctuation-dissipation relations are not satisfied [3, 4], it has been shown that some three-dimensional driven-dissipative systems close to a critical point may show emergent fluctuation-dissipation relations and, therefore, universal asymptotic thermalisation [5–15]. This, however, was suggested not to be the case for some two dimensional systems, where the dissipation has a more profound qualitative effect, completely destroying the analogous equilibrium order, and bringing the system to a different universality class [16].

This finding was of particular importance in the context of driven-dissipative two-dimensional bosonic superfluids, such as for example exciton-polaritons in semiconductor microcavities, where collective phase fluctuations were found to preclude the algebraic order in the thermodynamic limit, leading to a stretched exponential decay

of first order coherence characteristic of a Kardar-Parisi-Zhang phase (KPZ) [16]. Even if later estimates of the KPZ length scales for incoherently driven microcavities appeared to be beyond the reach of current experiments, and the presence of free vortices with screened repulsive interactions [17] might preclude the possibility of the KPZ phase [18, 19], the emerging order and the type of phase transition in these systems is still subject to an intense debate [20]. This is particularly true in light of the fact that exact stochastic simulations able to account for vortices [21], but also experiments [22], observed a clear transition from exponential decay of correlations to algebraic order but with an algebraic exponent α as large as four times the equilibrium upper bound, when approaching the BKT transition, suggesting an “over-shaken” but a superfluid state [21].

In this work, investigating parametrically driven polaritons, we show that the type of order adopted is in fact not an intrinsic property of the system but can be strongly sensitive to the driving process able to tune the system between two different universality classes by only a relatively small change in the driving strength. The key feature we exploit here is the spatial anisotropy that is imprinted on the system by the wave vector of the driving laser. In the long-wavelength theory for parametrically driven polaritons, which we derive in Sec. II, the effective degree of anisotropy is measured by a single parameter Γ . This quantity depends in a non-trivial way on the system parameters and, in particular, on the driving strength. In the region close to the optical parametric oscillator (OPO) upper threshold, but at lower powers, the system develops a steady state, which (if vortices remain bound) falls into the KPZ universality class [16, 23]

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with a non-equilibrium fixed point [16, 24] and no counterpart in equilibrium systems. However, by increasing the strength of the external drive towards the OPO upper threshold, the effective anisotropy crosses a critical value and the properties of the system are governed by an equilibrium fixed point. The system thus falls into the Edwards-Wilkinson (EW) universality class [25], which captures the universal properties of many different equilibrium systems, particularly the low temperature spin-wave theory of the XY model [1], exhibiting a BKT transition to off-diagonal algebraic order ensuring superfluidity. Note, that the equilibrium fixed point is approached for larger driving strengths than the non-equilibrium one, suggesting that we are not simply observing an approach to equilibrium as the external dissipation diminishes, but rather a more profound interplay between drive, dissipation and spatial anisotropy.

The various universal scaling regimes that can be accessed with polaritons were first discussed in the context of *incoherently* driven systems in Ref. [16]. However, while in both driving schemes the effective long-wavelength theory takes the form of the (anisotropic) KPZ equation, which is the origin of the rich universal behaviour, the underlying physics is completely different. In the case of incoherent pumping, the KPZ equation follows from a standard hydrodynamic description of the dynamics of the polariton fluid, and governs fluctuations of the phase of the condensate. On the other hand, coherent laser driving pins the condensate phase at the pump wave-vector, and the derivation of the long-wavelength theory is much more subtle: here, it is the *relative* phase of signal and idler modes, which are both macroscopically populated above the OPO threshold, that is free to fluctuate. This is the Goldstone mode that determines the physics at large scales. As a consequence of the different mechanisms leading to the KPZ equation, the natural scales of the coefficients appearing in this equation are vastly different in incoherently and coherently pumped polaritons. In particular, in the former case both the characteristic KPZ non-linearity and the degree of anisotropy are typically small [16], making the observation of novel non-equilibrium features challenging. Contrary to that, here we show that in OPO systems those quantities can be tuned over a wide range of values simply by changing the driving strength. Hence, we determine under which conditions these scaling regimes are feasible in current experiments on inorganic and organic microcavities. Moreover, we analyse the renormalisation group (RG) flow equations in a finite system, and estimate the relevant length scales by varying the control parameters such as the external laser drive, and the detuning between the cavity photons and the excitons.

The phenomena we discuss here are fundamentally induced by strong fluctuations, making the mean field approach, that has been applied in most of the existing literature on polaritons in the OPO regime [26–28], insufficient. Much rather, the RG techniques we use are tailored to address universal behaviour that occurs on

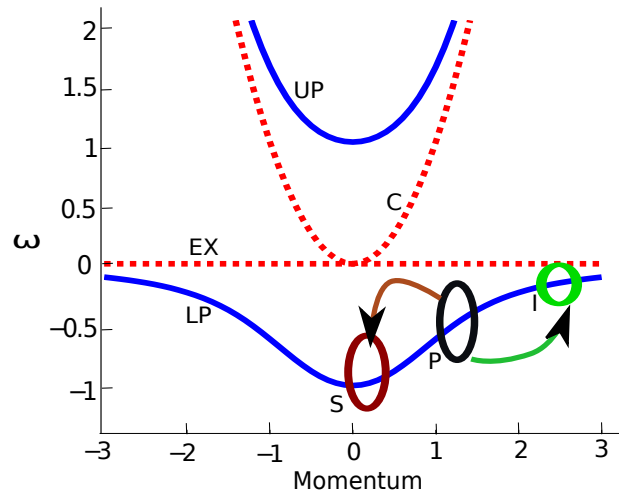


FIG. 1. **Polariton system.** Dispersions for the exciton (EX) and cavity photon (C) with zero detuning mixing to form upper (UP) and lower polaritons (LP). The external drive introduces pairs of lower polaritons with momentum \mathbf{k}_p and energy ω_p (P) which scatter into the signal (S) and idler (I) states while conserving energy and momentum.

large length and time scales, beyond what is accessible with exact numerical methods [21].

By establishing the universal regimes that are accessible with parametrically driven polaritons, we take a major step towards understanding of phases and phase transitions in 2D driven-dissipative systems. In particular, we show that the universal physics in OPO polaritons is much richer than anticipated, and has surprising connections to seemingly remote fields such as collective behaviour in active systems [24], thus opening a whole new perspective on coherently driven polaritons.

II. SYSTEM AND THEORETICAL DESCRIPTION

Exciton-polaritons are bosonic quasi-particles emerging in the regime of strong coupling between excitons in semiconductors and a cavity photon mode [2, 29] (see Fig. 1). The coherent mixing between light and matter excitations results in the emergence of two bands in these systems, termed the upper and lower polariton. Due to mirror imperfections the lifetime of photons and hence polaritons is finite, necessitating continuous external laser driving to maintain their finite population. In the stationary state resulting from the compensation of gain and losses, detailed balance is violated and the system is therefore not in thermal equilibrium.

The properties of this non-equilibrium stationary state are vitally influenced by the implementation of the laser driving. In particular, if the laser frequency is chosen to resonantly populate highly excited states, the polaritons generated in this way undergo complex scattering before condensing in the lower polariton band. All co-

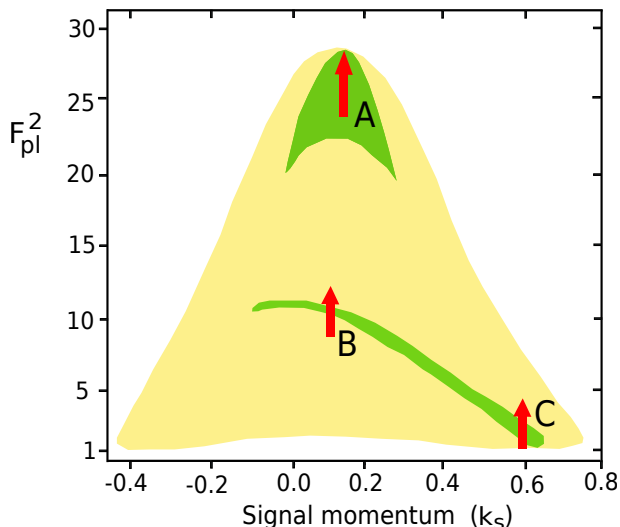


FIG. 2. **Stability diagram.** Regions of instability of the pump-only state towards an OPO solution (yellow and green). The simplest three-mode OPO state is stable for parameters marked by green, i.e. at k_s around 0.1 for high F_p^2 (region A and B) and over a more extended higher k_s range at low F_p^2 (region C). In the yellow region both the pump-only and the three mode ansatz are unstable. The red arrows indicate the direction of increasing pump power shown in Figs. 5, 4 and 6. Note that the pump intensity (vertical axis) is normalized to the lower threshold, $F_{pl}^2 \equiv (F_p/F_p^{\text{lo}})^2$. Parameters are as in the text with zero detuning and $k_p=1.4$.

herence of the exciting laser is lost in these processes and not transferred to the lower polariton states. The dynamics of the incipient condensate under such *incoherent* pumping is commonly described *phenomenologically* in terms of a generalized Gross-Pitaevskii equation [2]. In contrast to the incoherent pumping scheme, in the *coherent* scheme polaritons are excited with a monochromatic external laser acting resonantly on or close to the lower polariton dispersion [30]. The absence of complex scattering processes facilitates an *ab initio* rather than a phenomenological description as we detail below. As a consequence of coherent driving of lower polaritons, two symmetries which generically are present for incoherent pumping should be discussed in some more detail at this point: (i) The U(1) symmetry under rotations of the phase of the lower polariton field, and (ii) symmetry under spatial rotations.

The second point (ii) is addressed by specifying the coherent driving term for our problem, $f_p^* \psi_p + \text{h.c.}$ with external force $f_p = F_p e^{i(\mathbf{k}_p \cdot \mathbf{r} - \omega_p t)}$ and pump field ψ_p . Clearly, the directionality imprinted by the external driving field explicitly breaks the rotational symmetry of the problem on the microscopic level. It is a key point of this paper to elaborate on the consequences of this fact for the macroscopic observables.

Regarding (i), we need to specify the interaction term between the pump field ψ_p , and the signal and idler modes ψ_s, ψ_i , respectively. It describes the interconver-

sion of two pump field photons into a pair of signal and idler photons (see Fig. 1), $\sim \psi_i^* \psi_s^* \psi_p^2 + \text{h.c.}$. To begin with, we thus have three phases of pump, signal and idler fields. However, the external coherent pump term locks the phase of the pump field ψ_p via the coherent drive term, in turn locking the *sum* of phases of signal and idler fields ψ_s and ψ_i via the interaction term. On the other hand, their *difference* is not fixed by the dynamics of the system. Thus, there is one remaining U(1) phase rotation invariance left, which is generated by the transformation

$$\begin{pmatrix} \psi_s \\ \psi_i \\ \psi_p \end{pmatrix} \mapsto \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{-i\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_s \\ \psi_i \\ \psi_p \end{pmatrix}. \quad (1)$$

The residual U(1) symmetry introduced in this way can be broken spontaneously, and is responsible for the existence of a gapless phase mode above the OPO threshold [13, 31–33], which in turn governs the long distance coherence behaviour of the OPO condensate to be investigated in this paper.

More precisely, in a coherently driven system, depending on the strength of the external pump power, we distinguish two different regimes: The *pump-only* state, with only one mode, ψ_p , substantially occupied (white region in Fig. 2). Such a state is characterized by a momentum \mathbf{k}_p and frequency ω_p , which coincide with the momentum and frequency of the external pump. The phase of the pump-only state is locked by the external drive. However, in a certain pumping regime (green region in Fig. 2), pairs of polaritons scatter from the pump state into two new substantially occupied states, the *signal* ψ_s and the *idler* ψ_i , with momentum \mathbf{k}_s and \mathbf{k}_i and frequencies ω_s and ω_i , respectively (see Fig. 1). The scattering process is determined by the resonance conditions: $\mathbf{k}_s + \mathbf{k}_i = 2\mathbf{k}_p$ and $\omega_s + \omega_i = 2\omega_p$. Thus, in the OPO regime, the lower polariton field ψ_{LP} can be split into three contributions:

$$\psi_{\text{LP}}(\mathbf{r}, t) = \sum_{j=s,p,i} \psi_j(\mathbf{r}, t) e^{i(\mathbf{k}_j \cdot \mathbf{r} - \omega_j t)}. \quad (2)$$

In a mean-field treatment ignoring fluctuations, the amplitudes $\psi_j(\mathbf{r}, t)$ are spatially homogeneous and time-independent. Below we study the influence of fluctuations on the spatial and temporal coherence properties of the lower polariton field.

The yellow region in Fig. 2 marks the range of parameters in which polaritons are parametrically scattered to more than two additional momentum states (i.e., there are additional satellite states or the signal mode extends over a ring in a momentum space). However, here we focus on the regime in which the three-mode ansatz (2) is stable.

As a short digression, it is interesting to note which kind of order is established upon crossing the OPO threshold. Assuming that the amplitudes in Eq. (2) are constants which we write in density-phase representation

as $\psi_j = \sqrt{\rho_j} e^{i\phi_j}$, the density of lower polaritons is given by

$$|\psi_{\text{LP}}(\mathbf{r}, t)|^2 = \sum_j \rho_j^2 + 2 \left(\sqrt{\rho_s \rho_p} \cos(\phi_s - \phi_p + (\mathbf{k}_{si} \cdot \mathbf{r} - \omega_{si}t)/2) + \sqrt{\rho_p \rho_i} \cos(\phi_i - \phi_p - (\mathbf{k}_{si} \cdot \mathbf{r} - \omega_{si}t)/2) + \sqrt{\rho_s \rho_i} \cos(\phi_s - \phi_i + \mathbf{k}_{si} \cdot \mathbf{r} - \omega_{si}t) \right), \quad (3)$$

where we denote $\mathbf{k}_{si} = \mathbf{k}_s - \mathbf{k}_i$ and $\omega_{si} = \omega_s - \omega_i$, and we used the resonance conditions stated above. Thus, at the mean-field level, the OPO regime is characterized by density-wave and time-crystal order, with base wave vector and frequency $\mathbf{k}_{si}/2$ and $\omega_{si}/2$, respectively. The Goldstone mode alluded to above corresponds to fluctuations of the relative phase $\phi_s - \phi_i$. These can be incorporated in Eq. (3) by replacing $\phi_s \rightarrow \phi_s + \theta(\mathbf{r}, t)$ and $\phi_i \rightarrow \phi_i - \theta(\mathbf{r}, t)$ (the analysis of fluctuations is carried out systematically below), which shows that the Goldstone mode is encoded in the phase of the spatiotemporally periodic order. In particular, topological defects in $\theta(\mathbf{r}, t)$ are dislocations in the combined density wave and time crystal. However, by filtering the lower polariton field in momentum space as is usually done in experiments [34], it is possible to single out the signal and idler modes. Then, topological defects in $\theta(\mathbf{r}, t)$ appear as ordinary vortices. Using this approach, coherence of the OPO state can be quantified by measuring the two-point correlation function of the signal mode,

$$g^{(1)}(\mathbf{r}, t) = \langle \psi_s(\mathbf{r}, t) \psi_s^*(0, 0) \rangle. \quad (4)$$

In the following, we investigate how mean-field order is affected by fluctuations of the Goldstone mode.

A. Keldysh field integral approach

As pointed out above, for coherently pumped polaritons it is possible to derive a microscopic description. A convenient framework is provided by the Keldysh field integral formalism (see [4] for a recent review on applications to driven-dissipative systems). In this formalism, the coupled dynamics of excitons and photons under the influence of laser driving and cavity losses is encoded in a field integral [35] $Z = \int \mathcal{D}[\psi_s, \psi_i, \psi_p, \bar{\psi}_s, \bar{\psi}_i, \bar{\psi}_p] e^{iS_{\text{OPO}}}$, which we write here already in terms of the three polariton modes introduced in Eq. (2). Since we are interested in the long range dynamics of the system, we neglect quartic terms involving more than a single quantum field. This *semiclassical* approximation is applicable in the long-wavelength limit and can be justified formally by canonical power counting, which shows that the neglected terms are irrelevant in the renormalisation group

sense [36]. Thus, the action S_{OPO} becomes S_C :

$$S_C = \int dt d^2\mathbf{r} \left\{ -(f_p^* \psi_p^Q + f_p \bar{\psi}_p^Q) + \sum_{j=s,p,i} \left[\frac{1}{X_j^2} (\bar{\psi}_j^C \bar{\psi}_j^Q) \begin{pmatrix} 0 & [D_0^A]_j^{-1} \\ [D_0^R]_j^{-1} & [D_0^{-1}]_j^K \end{pmatrix} \begin{pmatrix} \psi_j^C \\ \psi_j^Q \end{pmatrix} - g_X \left((2(|\psi_s^C|^2 + |\psi_i^C|^2 + |\psi_p^C|^2) - |\psi_j^C|^2) \psi_j^C \bar{\psi}_j^Q + \text{c.c.} \right) \right] - g_X \left(2\psi_s^C \psi_i^C \bar{\psi}_p^C \bar{\psi}_p^Q + (\psi_p^C)^2 (\bar{\psi}_i^C \bar{\psi}_s^Q + \bar{\psi}_s^C \bar{\psi}_i^Q) + \text{c.c.} \right) \right\}, \quad (5)$$

where $\psi_{c,j}$ ($\psi_{q,j}$) is the classical (quantum) component of the field ψ_j . The inverse advanced Green's function reads $[D_0^A]_j^{-1} = i\partial_t + \omega_j - \omega_{\text{LP}}(\mathbf{k}_j - i\nabla) - i\gamma_j$, from which the inverse retarded Green's function can be obtained by using the relation $[D_0^R]_j^{-1} = ([D_0^A]_j^{-1})^\dagger$; γ_j is the decay rate of the mode j , and the lower polariton dispersion, in dimensionless units,¹ is given by

$$\omega_{\text{LP}}(\mathbf{q}) = \frac{1}{2} \left(q^2 + \delta_{CX} - \sqrt{(q^2 + \delta_{CX})^2 + 4} \right), \quad (6)$$

where δ_{CX} is the detuning between cavity photons and excitons. A typical dispersion relation for zero detuning is shown in Fig. 1. The Keldysh part of the inverse Green's function, $[D_0^{-1}]_j^K = i2\gamma_j$, stems from integrating out the bosonic decay bath fields in the Markovian approximation [35]. $X_j \equiv X(\mathbf{k}_j)$ is the excitonic Hopfield coefficient of the mode j with momentum \mathbf{k}_j [29], and g_X is the strength of the exciton-exciton interaction. Finally, the monochromatic external pump is, as mentioned previously, of the form $f_p = F_p e^{i(\mathbf{k}_p \cdot \mathbf{r} - \omega_p t)}$, where F_p is taken to be a positive real number. Since the action S_C is only quadratic in quantum fields we can make use of Martin-Siggia-Rose formalism [36–38] and map the functional integral to the set of coupled stochastic differential equations for the signal, idler, and pump modes, which determine the dynamics of the system:

$$\begin{aligned} i\partial_t \psi_s &= \Omega_s \psi_s - i\gamma_s + \tilde{g}_X \psi_p^2 \psi_i^* + \xi_s, \\ i\partial_t \psi_i &= \Omega_i \psi_i - i\gamma_i + \tilde{g}_X \psi_p^2 \psi_s^* + \xi_i, \\ i\partial_t \psi_p &= \Omega_p \psi_p - i\gamma_p + 2\tilde{g}_X \psi_s \psi_i \psi_p^* + f_p + \xi_p, \end{aligned} \quad (7)$$

where $\tilde{g}_X \equiv g_X X_p^2 X_i X_s$, $\psi_j \equiv \psi_j / X_j$, we use the shorthand notation [35] $\Omega_j \equiv -\omega_j + \omega_{\text{LP}}(\mathbf{k}_j - i\nabla) + g_X X_j^2 (2(\tilde{n}_s + \tilde{n}_p + \tilde{n}_i) - \tilde{n}_j)$ (with $\tilde{n}_j = X_j^2 |\psi_j|^2$) and

¹ Here and in the following we are using dimensionless units, measuring time, length, and energy in $2/\Omega_R$, $\sqrt{\hbar}/(\Omega_R m_C)$ and $\hbar\Omega_R/2$ respectively, where Ω_R is the Rabi frequency of the exciton-photon coupling, and m_C is the cavity photon effective mass.

we redefine the external pump $X_p f_p \rightarrow f_p$. The terms $\xi_{s,i,p}$ are Gaussian noise sources which have vanishing expectation value, $\langle \xi_j(\mathbf{r}, t) \rangle = 0$, and white spectrum, $\langle \xi_j(\mathbf{r}, t) \xi_{j'}^*(\mathbf{r}', t') \rangle = 2\gamma_j \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') \delta_{jj'}$. Note that the dynamical equations (7) are invariant under the U(1) transformation expressed in (1).

B. Long-wavelength theory in the OPO regime: mapping to the anisotropic KPZ equation

The stochastic equations (7) for the signal, idler, and pump modes provide a convenient starting point for deriving the effective long wavelength theory for polaritons in the OPO regime. We follow the usual strategy of parametrizing fluctuations around the mean-field solution in the density-phase representation, i.e., we write the three modes as

$$\psi_j(\mathbf{r}, t) = (\sqrt{\rho_j} + \pi_j(\mathbf{r}, t)) e^{i(\phi_j + \theta_j(\mathbf{r}, t))}, \quad (8)$$

where $\sqrt{\rho_j}$ and ϕ_j are the homogeneous and stationary mean-field density and phase, respectively, obtained by solving Eq. (7) with $\xi_{s,i,p} \equiv 0$; fluctuations around the mean-field solution are encoded in the fields $\pi_j(\mathbf{r}, t)$ and $\theta_j(\mathbf{r}, t)$. The key point that allows us to considerably simplify the equations resulting from inserting the ansatz (8) in Eq. (7) is that fluctuations of the relative phase $\theta(\mathbf{r}, t) = \theta_s(\mathbf{r}, t) - \theta_i(\mathbf{r}, t)$ of signal and idler modes, i.e., fluctuations of the Goldstone mode, are *gapless*, due to the U(1) symmetry expressed at (1), while fluctuations of all other are *gapped*. This implies that in the limit of long wavelength and low frequencies the latter fluctuations are small and the equations of motion can be linearised in these variables, which can then be eliminated. Details of this calculation are given in Appendix A. It results in a single stochastic equation for the Goldstone mode θ , which takes the form of the anisotropic KPZ (aKPZ) [24, 39] equation with an additional drift term proportional to $\nabla\theta$:

$$\partial_t \theta = \sum_{i=x,y} \left[D_i \partial_i^2 \theta + \frac{\lambda_i}{2} (\partial_i \theta)^2 \right] + \mathbf{B} \cdot \nabla \theta + \eta. \quad (9)$$

The coefficients $D_{x,y}$, $\lambda_{x,y}$ and \mathbf{B} result from linear combinations of different parameters appearing in Eqs. (7) (see Appendix A for details). The Gaussian white noise term η derives from the different noise terms ξ_j and satisfies $\langle \eta(\mathbf{r}, t) \rangle = 0$ and $\langle \eta(\mathbf{r}, t) \eta(\mathbf{r}', t') \rangle = 2\Delta \delta(\mathbf{r} - \mathbf{r}') \delta(t - t')$, where the noise strength Δ is related to the decay rates γ_j . In addition to eliminating massive fluctuations as discussed above, to obtain Eq. (9) we also expanded the lower polariton dispersion (6) around each mode j with momentum \mathbf{k}_j to second order in the gradient:

$$\omega_{\text{LP}}(\mathbf{k}_j - i\nabla) \approx \omega_{\text{LP}}(\mathbf{k}_j) - i\omega_{1j} \cdot \nabla - \nabla^T \omega_{2j} \nabla. \quad (10)$$

This expression shows clearly that the finite value of the pump wave vector \mathbf{k}_p (and hence of the idler wave

vector, and in some cases also of the signal wave vector) lies at the heart of the spatial anisotropy of the system, which leads in particular to the matrix ω_{2j} having two distinct eigenvalues, resulting in $D_x \neq D_y$ and $\lambda_x \neq \lambda_y$ in the effective long-wavelength description Eq. (9). This should be compared to incoherently pumped polaritons, for which the expansion of the lower polariton dispersion around zero momentum, $\omega_{\text{LP}}(-i\nabla) \approx \omega_{\text{LP}}(0) - \nabla^2/(2m_{\text{LP}})$, where m_{LP} is the mass of lower polaritons, leads to the *isotropic* KPZ equation [16].² Below we show how the values $D_{x,y}$ and $\lambda_{x,y}$, and hence the effective degree of anisotropy, depend on system parameters such as pumping and detuning. Crucially, by tuning these parameters we can access different universal scaling regimes of Eq. (9).

We note that the drift term $\mathbf{B} \cdot \nabla \theta$ can be eliminated from Eq. (9) by introducing a new variable $\theta'(\mathbf{r}, t) = \theta(\mathbf{r} + \mathbf{v}_0 t, t)$, i.e., by transforming to a frame of reference that moves at a velocity \mathbf{v}_0 . For $\mathbf{v}_0 = \mathbf{B}$, the equation of motion of θ' is given by the aKPZ equation without the drift term. It is thus sufficient to consider the latter equation, and transform back to the laboratory frame of reference only for calculating observables in terms of the original variable θ .

C. Scaling regimes of the anisotropic KPZ equation

In the previous section we showed that fluctuations around the three-mode OPO state (2) are governed by the anisotropic KPZ equation (9). What does this mean for the spatial and temporal coherence of the polariton condensate as measured by the first order coherence function Eq. (4)? There are three aspects which make the physics of Eq. (9) rich but also complex to analyse: (i) spatial anisotropy, (ii) the non-linear terms with coefficients $\lambda_{x,y}$, and (iii) the compactness of θ , which implies that this field can contain topological defects. Approaching the problem analytically, difficulties (ii) and (iii) can be controlled perturbatively, if both the non-linearities $\lambda_{x,y}$ and the vortex fugacity y , which is a measure of the probability of vortex-antivortex pairs forming at a microscopic distance, are small parameters.³ Then, as we describe in the following, depending on (i) the strength of anisotropy quantified by the anisotropy parameter

$$\Gamma \equiv \lambda_y D_x / (\lambda_x D_y), \quad (11)$$

based on the perturbative treatment we expect strikingly different behaviour in the weakly and strongly

² However, even in incoherently pumped polaritons some degree of anisotropy can be induced by the crystal structure and the splitting of transverse electric and transverse magnetic cavity modes [2, 40].

³ We note that y depends on the physics on short scales (information which is not contained in the long-wavelength description Eq. (9)) but could in principle be calculated from the stochastic equations (7).

anisotropic regimes, characterized by $\Gamma > 0$ and $\Gamma < 0$, respectively.

To understand why $\Gamma = 0$ separates these regimes, we first note that $D_{x,y} > 0$ is required for Eq. (9) to be dynamically stable; Therefore, $\Gamma < 0$ corresponds to λ_x and λ_y having opposite signs. If λ_x and λ_y have the same sign and hence $\Gamma > 0$, it does not make a difference whether $\lambda_{x,y}$ are both positive or negative. In fact, the former case is related to the latter by the transformation $\theta \rightarrow -\theta$ in Eq. (9) (after the drift term has been removed as described above). Thus, the physics can change *qualitatively* only when $\lambda_{x,y}$ have opposite sign and thus $\Gamma < 0$. This is indeed found to be the case in the RG analysis.

In the weakly anisotropic (WA) regime both the non-linear terms $\lambda_{x,y}$ [24, 39] and the fugacity y [19] are relevant couplings, i.e., they grow under renormalisation. In the absence of vortices this would imply that the correlation function $g^{(1)}(\mathbf{r}, t)$ takes the form of a stretched exponential with KPZ scaling exponents (see Eq. (14) below). This behaviour would be observable on length and time scales greater than L_{KPZ} and t_{KPZ} , respectively, which mark the breakdown of the perturbative treatment in $\lambda_{x,y}$. However, eventually vortices might unbind at a scale L_v and after a time t_v [19], leading to exponential decay of correlations (and a absence of superfluid behaviour) beyond these scales. For a detailed discussion of the influence of vortices in this regime see Appendix B.

The physics is quite different in the strongly anisotropic (SA) regime: for $\Gamma < 0$, the non-linearities $\lambda_{x,y}$ are irrelevant and flow to zero. Then, the linearised version of Eq. (9) exhibits a BKT transition driven by the noise strength (which in turn depends on the loss rates and the external drive, see Appendix A), i.e., at low noise a superfluid phase with algebraic order is possible [16, 24] even in thermodynamic limit of an infinite system. Intriguingly, OPO polaritons allow to cross the boundary between the WA and SA regimes. This is shown in the next section. Before that, in the remainder of this section, we discuss in detail the renormalisation group (RG) flow and the two scaling regimes of the aKPZ equation.

The RG flow of the aKPZ equation in the absence of vortices was analysed in Refs. [24, 39]. It can be parameterized in terms of only two independent quantities, the anisotropy parameter Γ introduced in Eq. (11), and the rescaled dimensionless non-linearity g which is defined as

$$g \equiv \lambda_x^2 \Delta / (D_x^2 \sqrt{D_x D_y}). \quad (12)$$

To leading order in g , the RG flow equations read:

$$\begin{aligned} \frac{dg}{dl} &= \frac{g^2}{32\pi} (\Gamma^2 + 4\Gamma - 1), \\ \frac{d\Gamma}{dl} &= \frac{\Gamma g}{32\pi} (1 - \Gamma^2). \end{aligned} \quad (13)$$

As described above, depending on the value of Γ , we distinguish between WA and SA regimes. In the former

case, λ_x and λ_y have the same sign (the coefficients $D_{x,y}$ have to be positive to ensure stability). Then, the non-linearity is marginally relevant, and the RG flow takes the system to a strong coupling fixed point at g_* which is beyond the scope of the perturbative treatment. Moreover, $\Gamma \rightarrow 1$, i.e., at large scales rotational symmetry is restored⁴ and thus the system falls into the usual isotropic KPZ universality class.

On the other hand, in the SA regime with $\Gamma < 0$, which is realized when the coefficients λ_x and λ_y have opposite sign, the non-linearity g is irrelevant and flows to zero. As a consequence, the aKPZ equation becomes a linear stochastic differential equation, which is governed by an equilibrium fixed point at $g = 0, \Gamma = -1$, and the system falls into the EW universality class [25].

Having discussed how the RG flow of the aKPZ equation is structured by different fixed points in the WA and SA regimes, it is natural to ask for observable consequences of these findings. Universal scaling behaviour leaves its mark in the long-time and long-range decay of correlations. Hence, in the following we discuss the form of the correlation function (4) implied by these results, and which modifications are to be expected due to the possible occurrence of vortices.

1. Weakly anisotropic regime

In the WA regime, the two point correlation function (4) $g^{(1)}(\mathbf{r}, t) \propto e^{-C(\mathbf{r}, t)/2}$ (this form assumes that density fluctuations are negligible as compared to fluctuations of the Goldstone mode, see Sec. II B), where $C(\mathbf{r}, t) = \langle (\theta(\mathbf{r}, t) - \theta(0, 0))^2 \rangle$, shows a stretched exponential decay [41]:

$$C(\mathbf{r}, t) \sim \tilde{r}^{2\chi} F_{\text{KPZ}}(c_1 t / \tilde{r}^z) \sim \begin{cases} \tilde{r}^{2\chi} & \text{for } \tilde{r}^z \gg c_1 t, \\ t^{2\chi/z} & \text{for } c_1 t \gg \tilde{r}^z, \end{cases} \quad (14)$$

where $\tilde{r}^2 = (x/x_0)^2 + (y/y_0)^2$ encodes the anisotropy of the system and the parameter c_1 depends on the microscopic parameters. The limiting forms follow from the asymptotic behaviour of the scaling function, $F_{\text{KPZ}}(w) \sim A_1$ for $w \rightarrow 0$ and $F_{\text{KPZ}}(w) \sim A_2 w^{2\chi/z}$ for $w \rightarrow \infty$, with non-universal constants A_1 and A_2 . In two spatial dimensions, the roughness exponent is $\chi \approx 0.39$ (see Refs. [42, 43] for recent numerical investigations of KPZ scaling and [44] for a functional RG analysis), and the dynamical exponent z can be obtained from the exact scaling relation $\chi + z = 2$.

The scaling form Eq. (14) applies to the co-moving reference frame (see the discussion below Eq. (9)), in which the drift term $\mathbf{B} \cdot \nabla \theta$ is absent. We can calculate $g^{(1)}(\mathbf{r}, t)$

⁴ For $\Gamma = 1$, by rescaling the units of length Eq. (9) can be brought to isotropic form with $D_x = D_y$ and $\lambda_x = \lambda_y$.

in the original frame simply by replacing $\mathbf{r} \mapsto \mathbf{r} + \mathbf{B}t$ in Eq. (14), which yields

$$C(\mathbf{r}, t) \sim |\mathbf{r} + \mathbf{B}t|^{2\chi} F_{\text{KPZ}}(c_1 t / |\mathbf{r} + \mathbf{B}t|^z). \quad (15)$$

From this expression we explore the consequences of a non-vanishing drift term \mathbf{B} on the correlations of the system for the KPZ scaling. In particular, we find for spatial correlations at equal times

$$C'(\mathbf{r}, 0) \sim \tilde{r}^{2\chi}, \quad (16)$$

which coincides with the result for the case of vanishing drift term. However, temporal correlations are modified:

$$C'(0, t) \sim (Bt)^{2\chi} F_{\text{KPZ}}(c_1 t / (Bt)^z) \sim \begin{cases} t^{2\chi/z} & \text{for } t \ll \tau_c, \\ t^{2\chi} & \text{for } t \gg \tau_c. \end{cases} \quad (17)$$

Hence, the system exhibits two different exponents, depending on the time scale. Initially, the correlator $g^{(1)}(0, t)$ shows stretched exponential decay with exponent $2\chi/z$ characteristic of KPZ scaling. At longer times, the drift term causes the exponent to increase to 2χ , resulting in a faster decay of temporal correlations. The crossover time τ_c at which the transition between the two regimes occurs can be obtained from: $c_1 \tau_c \sim (B\tau_c)^z$, leading to $\tau_c \sim (B^z/c_1)^{1/(1-z)}$. The scaling forms (16) and (17) are approached on certain length and time scales. For small KPZ non-linearity g , the scale L_{KPZ} above which spatial correlations are expected to behave as (16) can in the isotropic case be estimated as [16]

$$L_{\text{KPZ}} = \xi_0 e^{8\pi/g}, \quad (18)$$

and the corresponding time scale, after which scaling behaviour according to Eq. (17) sets in, follows from diffusive scaling and is given by [19] $t_{\text{KPZ}} = L_{\text{KPZ}}^2 / \bar{D}$.⁵ In Eq. (18), $\xi_0 = \hbar / \sqrt{2m_{\text{LP}} g_X \sqrt{n_s n_i}}$ is the healing length of the system [45].

Finally, as we mentioned previously, taking into account the compactness of the phase in the KPZ equation in the WA regime, vortices have been predicted to unbind at a scale L_v [19]

$$L_v = \bar{\xi}_0 e^{2\bar{D}/\bar{\lambda}}, \quad (19)$$

leading to exponential decay of correlations beyond.⁶ Thus the algebraic or KPZ orders in the WA regime

might appear only as a finite size or transient phenomena. We refer the reader to Appendix B for a detailed description of the physics of the vortices in the WA regime of the compact KPZ equation.

2. Strongly anisotropic regime

In the SA regime the RG flow equations (13) approach the fixed point at $g = 0, \Gamma = -1$, belonging to the EW universality class [25]. Then, for a zero drift term \mathbf{B} , the correlations decay as power laws both in space and time [46–48]:

$$g^{(1)}(\mathbf{r}, t) \sim \tilde{r}^{-\alpha} F_{\text{EW}}(\tilde{r}^{z'}/(c_2 t)) \sim \begin{cases} \tilde{r}^{-\alpha} & \text{for } \tilde{r}^{z'} \gg c_2 t, \\ t^{-\alpha/z'} & \text{for } c_2 t \gg \tilde{r}^{z'}, \end{cases} \quad (20)$$

where the parameter c_2 depends on microscopic parameters. The scaling function $F_{\text{EW}}(w)$ behaves asymptotically as $F_{\text{EW}}(w) \sim A'_1$ for $w \rightarrow \infty$ and as $F_{\text{EW}}(w) \sim A'_2 w^{\alpha/z'}$ for $w \rightarrow 0$, where A'_1 and A'_2 are non-universal constants. The exponents are $z' = 2$ and $\alpha = \kappa(\infty)/(4\pi)$, with the renormalized scaled noise

$$\kappa(l) = \Delta(l) / \sqrt{D_x(l) D_y(l)}, \quad (21)$$

evaluated from the RG flow equations for the aKPZ equation in the limit $l \rightarrow \infty$ [24]. We obtain the correlations in the original frame of reference by reverting the coordinate transformation from the co-moving frame in which the drift term in Eq. (9) is absent. Replacing $\mathbf{r} \mapsto \mathbf{r} + \mathbf{B}t$ in Eq. (20), yields

$$g^{(1)}(\mathbf{r}, t) \sim |\mathbf{r} + \mathbf{B}t|^{-\alpha} F_{\text{EW}}(|\mathbf{r} + \mathbf{B}t|^{z'}/(c_2 t)). \quad (22)$$

We examine the consequences of a non-vanishing drift term \mathbf{B} on the correlations of the system. In particular, the spatial correlations at equal times behave as

$$g^{(1)}(\mathbf{r}, 0) \sim \tilde{r}^{-\alpha}, \quad (23)$$

which coincides with the result for the case of vanishing drift term. However, as in the KPZ scaling regime, temporal correlations are modified:

$$g^{(1)}(0, t) \sim (Bt)^{-\alpha} F_{\text{EW}}((Bt)^{z'}/(c_2 t)) \sim \begin{cases} t^{-\frac{\alpha}{z'}} & \text{for } t \ll \tau'_c, \\ t^{-\alpha} & \text{for } t \gg \tau'_c. \end{cases} \quad (24)$$

Thus, as in the WA regime, the system shows two different exponents, depending on the time scale. Initially, the correlator $g^{(1)}(0, t)$ shows algebraic decay with the characteristic $-\alpha/z'$ exponent. At longer times, however, the drift term causes the exponent to decrease to $-\alpha$, resulting in a faster decay of temporal correlations. The crossover time at which the transition between the two regimes occurs at $\tau'_c \sim (B^{z'}/c_2)^{1/(1-z')}$.

We should note that the *closed* 2D bosonic system in thermal equilibrium, in the absence of drive and dissipation, reveals a slightly different behaviour. In such a

⁵ While these estimates were originally derived for isotropic systems, we expect them to remain valid in the WA regime and adapt the expression for t_{KPZ} to the latter by replacing the isotropic diffusion constant by the geometric mean $\bar{D} = \sqrt{D_x D_y}$.

⁶ As in the above estimates of the characteristic KPZ scales we expect to obtain a valid estimate throughout the WA regime by replacing the isotropic diffusion constant and non-linearity by the geometric averages \bar{D} and $\bar{\lambda} = \sqrt{\lambda_x \lambda_y}$. Note that $\bar{\xi}_0$ is related to the vortex mobility and may differ substantially from ξ_0 , see Appendix B.

case the phase fluctuations obey the following equation: $\partial_t^2 \theta = D'_x \partial_x^2 \theta + D'_y \partial_y^2 \theta$, where D'_x, D'_y are the squares of the x and y -component, respectively, of the speed of sound [16, 37]. In such a case, the two point correlation function shows an algebraic order with the same exponent α for both space and time: $g^{(1)}(\mathbf{r}, t) \rightarrow \tilde{r}^{-\alpha}, t^{-\alpha}$ respectively, with $\alpha > 0$ [48]. This is a consequence of the linear dispersion of the gapless Bogoliubov excitation in $k, \omega \rightarrow 0$ limit.

Including the compactness of the phase in the SA regime does not preclude algebraic order and superfluidity. It leads to a well-known BKT [24] transition between a quasi-ordered and a disordered phases mediated by the binding/unbinding of vortices. The system falls into the XY universality class, which is the extension of the EW universality class for compact variables. We can estimate the phase boundary for algebraic order by considering a simple argument presented in Ref. [16, 24]: we assume that vortices only become relevant at scales where g has flowed to nearly 0, and hence we can use the RG flow equations (13) even though they do not include vortices.⁷ In this scenario, the BKT transition is estimated to occur at $\kappa(\infty) = \pi$, where $\kappa(\infty)$ is the renormalized scaled noise (21) in the limit $l \rightarrow \infty$. This condition defines the phase boundary $\kappa_0 = \kappa_*$ between ordered and disordered phases which reads:

$$\kappa_* = -\frac{4\pi\Gamma_0}{(1 - \Gamma_0)^2}, \quad (25)$$

where we have used the expression of $\kappa(\infty)$ as a function of the bare parameters κ_0 and Γ_0 [24]. Thus, when $\kappa_0 < \kappa_*$, the system shows algebraic order, whereas if $\kappa_0 > \kappa_*$ the algebraic order is destroyed by vortices resulting in exponential decay of correlations.

According to the above discussion, while incoherently pumped (and thus at best weakly anisotropic) 2D polaritons (or other photonic) systems are always disordered in the thermodynamic limit of infinite system size, and algebraic order or superfluidity can only be a finite size effect, the parametrically pumped polaritons are fundamentally different. The pumping process, which can be chosen at any wave-vector, can result in a high level of effective anisotropy, which allows us to enter the SA regime, governed by the XY equilibrium fixed point, thus ensuring algebraic order up to infinite distances in the thermodynamic limit. Such a high level of anisotropy would not be achievable by a crystal growth engineering aimed at creating different effective masses in perpendicular directions. Moreover, as we show below, the anisotropy can be changed simply by tuning experimental parameters

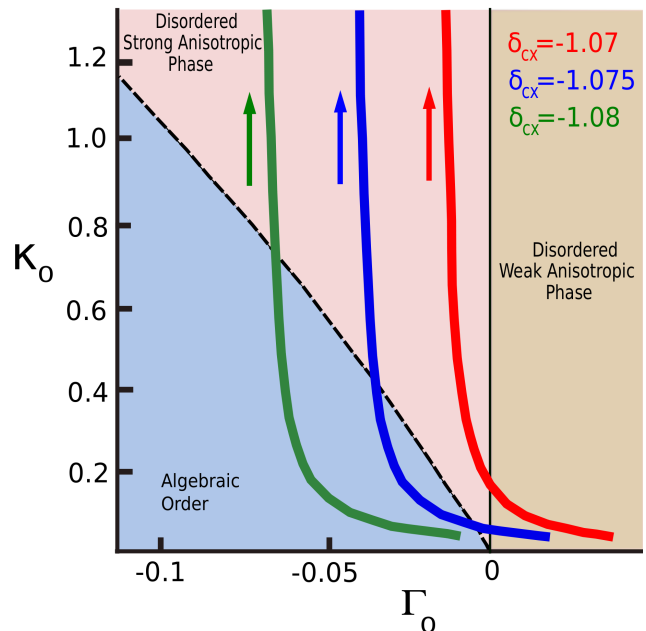


FIG. 3. **Crossover between non-equilibrium and equilibrium-like universal regimes.** Stable three-mode OPO configurations in the $\Gamma_0 - \kappa_0$ space for three different detunings $\delta_{CX} = -1.07$ (green), -1.075 (blue), -1.08 (red), $k_p = 1.4$ and $k_s = 0.1$ (region A in Fig. 2). The arrows indicate the direction of increasing the external drive strength. The dashed line shows the BKT phase boundary (see Eq. (25)) between algebraically ordered and the disordered phases. By increasing the external pump power, for $\delta_{CX} = -1.07$ we cross from the non-equilibrium (WA) to the equilibrium-like (SA) disordered regimes, for $\delta_{CX} = -1.075$ the systems shows *reentrance*, it starts in the WA, enters the SA algebraically ordered regime and finally goes back to the disordered but now SA regime. For $\delta_{CX} = -1.08$ we are in the SA equilibrium-like regime for all pump powers, and the system undergoes a BKT transition from an algebraically ordered to a disordered phase by increasing F_p .

such as the pump power or the detuning between the excitons and photons, allowing us to easily in one experiment move between different regimes.

III. EXPLORING SCALING REGIMES OF OPO POLARITONS

In this section we show that the OPO-polariton system can be driven from the non-equilibrium WA to an equilibrium-like SA regime by simply tuning the strength of the external pump power and the detuning between the cavity photons and excitons. We then consider implications for finite size systems. In general, polaritons in the OPO regime (or in the incoherently pumped scenario above condensation threshold) are characterised by a high degree of coherence. This is because the system size in experiments, due to intrinsic disorder in the samples limiting the spatial extent of useful regions, is rela-

⁷ We note, however, that even in the SA regime the non-linearities might induce screening of the vortex interaction with a screening length that is shorter than the scale at which $g \approx 0$. It is an interesting question for future research whether this affects the BKT transition.

tivity small in comparison to the relevant length scales of the decay of correlations, especially well above threshold where most experiments operate. Indeed, non-decaying spatial coherence, characteristic of BEC in 3D, was seen in most cases [49, 50], and even observation of the algebraic decay appeared challenging [22, 51, 52]. In order to minimise the influence of the finite size, which masks the underlying physics, we need to focus on samples and regimes, in which we have appreciable decay of coherence for the considered system size. In general, this would correspond to what we call *bad* samples, where the influence of dissipation is substantial but not strong enough to completely wash out any collective effects. In our opinion, the most promising microcavities are those used in early days of work on polariton condensation, where collective effects were already seen but the polariton lifetime and the Rabi splitting were quite small by current standards. Thus, we first focus on what we call a *bad* inorganic microcavity [21, 34], and in Sec. III E we compare this with better quality samples, characterised by longer lifetimes and larger Rabi splitting, used by most groups today, as well as with organic microcavities.

The *bad* microcavity is characterized by the following set of parameters: $\hbar\Omega_R = 4.4$ meV, $\hbar\gamma_j = 0.1$ meV (corresponding to a lifetime of 6.6 ps); $m_C = 2.5 \cdot 10^{-5}m_0$, $g_X = 2\mu\text{eV}\mu\text{m}^{-2}$. We choose the pump wave vector close to the inflection point of the lower polariton dispersion, $k_p = 1.4$, which corresponds to $1.61 \mu\text{m}^{-1}$ in dimensional units for the *bad* cavity, and $\omega_p = \omega_{\text{LP}}(k_p)$, as shown in Fig. 2. At the mean-field level, the system exhibits upper and lower thresholds for the OPO transition at pump powers indicated by F_p^{lo} and F_p^{up} respectively. Solving numerically exactly the analogue of Eq. (7) for the exciton-photon model and the same set of parameters with zero-detuning shows that the system undergoes a BKT-type phase transition at a pump power $F_p^{\text{BKT,lo}} \approx 1.014F_p^{\text{lo}}$ and $F_p^{\text{BKT,up}} \approx 0.999F_p^{\text{up}}$ [21]. The value of k_s is not determined by the three-modes ansatz (7) [53]. However, the stability analysis shown in Fig. 2 suggests a value of $k_s \approx 0.11 \mu\text{m}^{-1}$ for intermediate and high values of F_p , whereas the system chooses bigger values of k_s when approaching the lower threshold, i.e., $k_s \in [0.11 \mu\text{m}^{-1}, 0.7 \mu\text{m}^{-1}]$ (region C in Fig. 2).

A. Infinite system: crossover between weakly and strongly anisotropic regimes

We first focus on the region with small k_s (A and B in Fig. 2). The analysis of the anisotropy parameter Γ for the *bad* cavity shows that the system falls into the WA regime, i.e., $\Gamma > 0$, at all pump powers when the detuning between the photons and the excitons fulfils $-1.07 < \delta_{CX}$. The expected scaling behaviour of correlations is discussed in Sec. II C 1. In particular, algebraic decay of correlations is ruled out in this regime. However, this picture changes drastically for lower values of the detuning. When $\delta_{CX} \leq -1.07$, we enter the SA

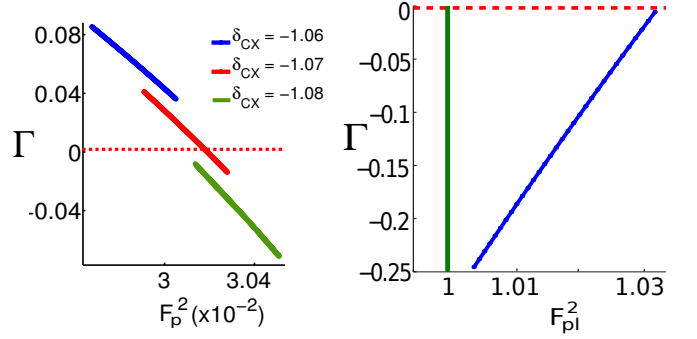


FIG. 4. **Changing the effective anisotropy by tuning the driving strength.** *Left panel:* Anisotropy parameter Γ as a function of pump power F_p^2 (in the same range as region A in Fig. 2) for different detunings $\delta_{CX} = -1.06$ (blue), -1.07 (red), -1.08 (green) in the stable three-mode OPO regime. Parameters are as in Fig. 3. For $\delta_{CX} = -1.075$ the anisotropy parameter crosses zero (dotted red line). Note that the extent of region A, i.e., the range of pump powers for which the three-mode OPO ansatz is stable, depends on the value of the detuning. *Right panel:* Anisotropy parameter Γ as a function of normalised pump power, where $F_{pl} \equiv F_p/F_p^{\text{lo}}$, at zero detuning but close to the lower OPO threshold, indicated by the green vertical line. The pump and signal have large momenta $k_p \approx 1.84$ and $k_s = 1.0$. The dotted red line marks the $\Gamma = 0$ line. This case corresponds to region C in Fig. 2.

regime, which is expected to have long-range properties which are qualitatively similar to an equilibrium system (see Sec. II C 2). For example, algebraic order can be destroyed by the proliferation of vortices as in the equilibrium BKT transition when the level of the effective noise is large, which is the case when the signal density is low. In Fig. 3 we show three characteristic cases of different detuning superimposed on the phase diagram generated by the relation (25) in the $\Gamma_0 - \kappa_0$ phase-space: i) For $\delta_{CX} = -1.07$ the system is always disordered. It crosses from the disordered non-equilibrium (WA) to the disordered equilibrium-like (SA) regime since $\kappa_0 > \kappa_*$ for all values of F_p . ii) $\delta_{CX} = -1.075$ is the most interesting case. The system shows reentrant behaviour [16]: by increasing the pump power we move from the disordered WA to the disordered SA regime, then by increasing the pump power further we reach the BKT phase transition to the algebraically ordered phase, followed by a second BKT transition close to the OPO upper threshold back to the SA disordered phase. iii) For $\delta_{CX} = -1.08$ the stable three-mode solutions lie in the SA regime for all pump powers. For smaller values of F_p the system is in the algebraically ordered phase, and it undergoes a BKT phase transition to a disordered phase by increasing F_p . As can be seen in the left panel of Fig. 4, all these three different cases show a nearly linear dependence of Γ on the intensity F_p^2 of the external drive.

Increasing detuning in the region close to the upper threshold (region A in Fig. 2) is one way to introduce a sufficient level of anisotropy to cross to the equilibrium-

like phase. However, there is also another source of anisotropy: the system can be driven to the SA regime by increasing the pump momentum, k_p , leading to an increase of the signal momentum, k_s , by tuning the pump power close to the lower OPO threshold (region C in Fig. 2). For example, for the *bad* cavity parameters with $\delta_{CX} = 0$ and $k_p = 2.11 \mu\text{m}^{-1}$ (1.84 in dimensionless units) we enter the SA regime for $k_s \approx 0.7 \mu\text{m}^{-1}$, as can be seen in the right panel of Fig. 4.

B. Finite system: the length scales of the weakly anisotropic regime

In this subsection, based on Eqs. (18) and (19), we estimate the relevant length scales of the WA regime to examine which phases can be seen in current semiconductor microcavities, and whether finite-size effects would hamper the underlying universal physics. We address, in particular, whether the non-equilibrium ordered KPZ phase can ever be seen in semiconductor microcavities. We focus here on our *bad* cavity parameters as the most promising to explore various phases.

We first explore the WA regime close to the upper threshold (region A in Fig. 2). The parameters of the aKPZ equation for this regime are shown in Fig. 10 in Appendix A. Most of the parameters are approximately constant as a function of the external pump strength, F_p . The drift term is non-zero only in the direction of the pump wave vector \mathbf{k}_p , which we have chosen to be along the x -axis. The dimensionless non-linearity g and the noise strength Δ asymptote to high values, which leads to a small value of L_{KPZ} (see Eq. (18)), only very close to the upper threshold. As can be seen in Fig. 5, even for the most promising *bad* cavity parameters L_{KPZ} is astronomically large at any reasonable distance from the upper threshold. It goes down to $100 \mu\text{m}$, currently the upper bound for any experiments, only at around 0.999 of the upper threshold (see right panel of Fig. 5). However, this point is already above the BKT transition (green dashed line in Fig. 5), where proliferating vortex-antivortex pairs destroy the KPZ scaling. Additionally, $L_v \ll L_{\text{KPZ}}$ for all values of F_p apart from those close to the upper threshold already beyond the BKT transition. (Also, note that as explained in Sec. II C 1 close to the upper threshold where fluctuations are strong the scale at which vortices unbind should be strongly renormalized and smaller than L_v in Eq. (19).) Thus, we conclude that in this regime (region A in Fig. 2) KPZ scaling would either be overshadowed by the algebraic order at scales below L_v due to the astronomically large length scales required, or destroyed by the BKT vortices resulting in an equilibrium-like behaviour in a finite system. The question remains: is there then no hope for the KPZ phase in microcavities in two dimensions and we are only left with equilibrium-like behaviour? We address this in the next section.

We should also comment that L_v drops down to less

than $100 \mu\text{m}$ for our *bad* cavity for some pump powers away from the BKT threshold — a scale which is quite realistic. This suggests that in such a case free vortices (not of BKT type) should destroy the algebraic order beyond this scale [19] (see Appendix B). However, exact simulations of stochastic dynamics for systems as large as $1000 \mu\text{m}$ [21] do not show any signs of this phase, even at very long times where a steady-state has clearly been reached, suggesting suppressed activation by e.g. an extremely small vortex mobility (Ref. [19] assumed instead a vortex mobility of the order of other scales in the problem), or that attractive interactions between vortex and antivortex at small distances may in reality prevent the free vortices from becoming relevant. It may also be that the rough formula for L_v underestimates the real value.

C. Finite system: searching for the KPZ phase

Unlike the incoherently pumped case, our OPO system offers more possibilities for parameter tuning. Interestingly, as we can see in Fig. 6, the system shows a regime at intermediate pump powers, $3.1 < (F_p/F_p^{\text{lo}})^2 < 3.5$, where g becomes large and so L_{KPZ} is small (region B in Fig. 2). In fact, in some parts of this range $g > 1$, meaning that the KPZ phase is expected at all length scales beyond the healing length. Such a regime does not exist for incoherently driven microcavities. Its presence in the OPO configuration is due to the non-monotonic behaviour of KPZ parameters as a function of pump power, associated with underlying instabilities towards more complex spatial patterns in the system such as the satellites formation or ring OPOs [53]. However, we also find that $L_v \approx \xi_0$ (see right panel Fig. 6) and therefore in principle the vortex phase could destroy the KPZ physics. But, due to the suppressed activation observed in numerical studies, the free vortices may never appear. Testing the possibility of the KPZ phase in the middle of the OPO region using the exact stochastic dynamics, and hopefully experiments, would give the final word on this. Note, that the only experiment measuring spatial coherence in the OPO configuration focuses on a different regime of powers [50]. Our estimates show that using pump powers a few times the OPO threshold in good quality samples as far as spatial disorder is concerned, but with relatively short polariton lifetime, is the most promising regime to observe signatures of the KPZ physics.

D. Finite system: crossover between weakly and strongly anisotropic regimes

In Sec. III A we have shown that the infinite polariton system can be tuned between two different universality classes (non-equilibrium KPZ and equilibrium EW), with completely different large-scale behaviours, by changing the exciton-photon detuning, and the properties of the drive such as the pump power, F_p , and wave-vector k_p

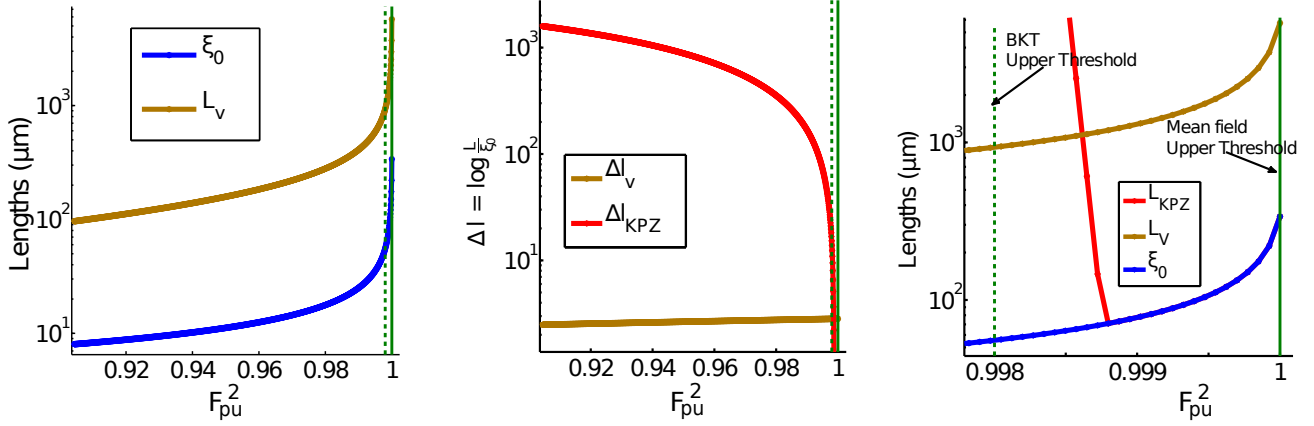


FIG. 5. **Characteristic length scales in the WA regime.** *Left and right panels:* Healing length ξ_0 (blue), length scale for the vortex-dominated disordered phase L_v (brown), and the KPZ length scale L_{KPZ} (red) as a function of the pump strength (normalised to the upper threshold, $F_{pu} \equiv F_p/F_p^{up}$). Vertical dotted (green) line indicates the BKT transition where V-AV pairs proliferate (taken from Ref. [21]) and the vertical solid (green) line the mean-field OPO threshold. Parameters as those discussed in the text for the *bad* microcavity at zero detuning $\delta_{CX} = 0$, $k_p = 1.4$ and $k_s = 0.1$. The two different panels correspond to two different ranges in pump strength. *Central panel:* logarithmic RG scale for the vortex-dominated phase (Δl_v) and for the KPZ-phase (Δl_{KPZ}) as a function of pump strength normalised to the pump at the upper threshold (vertical solid green line). Vertical dotted green line indicates the BKT transition. Note that L_v is expected to be strongly renormalized close to the threshold as explained in Appendix B.

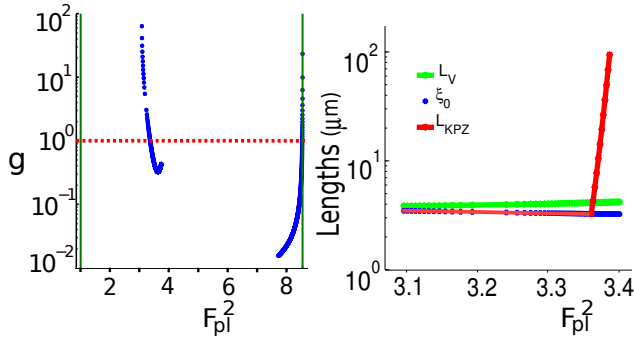


FIG. 6. **Characteristic length scales in the WA regime at intermediate pump powers.** *Left panel:* Non-linear parameter g as a function of the pump strength normalised to the pump at the lower threshold for stable three-mode OPO solution with zero detuning. The vertical green lines indicate the lower and upper mean field thresholds, and the red dotted line marks the $g = 1$ border. These are the stable solutions at intermediate values of F_p (region B in Fig. 2). *Right panel:* Characteristic length scales at the intermediate F_p shown in the left panel. Healing length ξ_0 (blue), length scale for vortex dominated disordered phase L_v (green) and the KPZ length scale L_{KPZ} (red) as a function of pump strength normalised to the pump at the upper threshold. Note that to the left of the nearly vertical increase of L_{KPZ} its value is indistinguishable from the healing length.

— easily realisable in current experiments. Here, we consider how this crossover is affected by the finite size.

We consider the case with detuning $\delta_{CX} = -1.07$ close to the upper OPO threshold (region A in Fig. 2). As we discussed, for this detuning the system can move from the

non-equilibrium (WA) to an equilibrium-like (SA) regime by increasing the external pump power. If the system is infinite, for these parameters, it is in the disordered phase, characterised by the exponential decay of correlations, in both regimes. However, the system may show algebraic order up to certain length scale L_{BKT} in the SA regime or L_v, L_{KPZ} in the WA regime. In the first case, this would require $\kappa_* < \kappa(l) < \pi$ (see (25)), and $L_{BKT} \equiv \xi_0 e^{l_{BKT}}$ is obtained by considering the 'BKT-transition criterion': $\kappa(l_{BKT}) \sim \pi$. We calculate L_{BKT} following Ref. [24] and the RG flow-equations (13), and the results are displayed in Fig. 7. We obtain that L_{BKT} takes reasonable physical values for $\Gamma \approx -0.0145$. When $\Gamma > -0.0145$, $L_{BKT} \rightarrow \infty$ which indicates that the algebraic order appears at all realistic physical length scales; whereas when $\Gamma < -0.0145$, $L_{BKT} \sim \xi_0$, and so the system is in a disordered phase at all length scales beyond the healing length.

Considering this, we find that there are two interesting scenarios at intermediate length scales when driving the system from the WA to the SA regime, as indicated by double arrows A and B in Fig. 7. The first case, arrow A, shows a transition between the disordered phase in the non-equilibrium WA regime to the algebraically ordered phase in the equilibrium-like SA regime by increasing the pump power and, consequently, crossing $\Gamma = 0$. This phenomenon appears for L such that $L_v < L < L_{BKT}$. In the second scenario (arrow B in Fig. 7), the system can be driven from the WA to the SA regimes without changing the phase, i.e. maintaining the algebraic order in both cases. This situation occurs for L fulfilling $L < L_v, L_{BKT}$. Scenario A requires length scales of the order

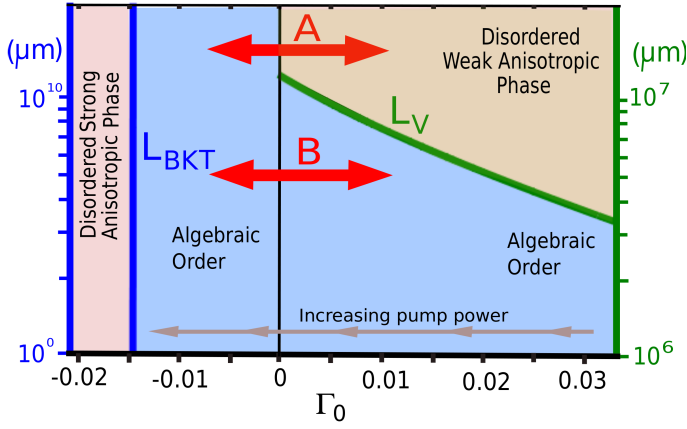


FIG. 7. **Crossover between SA (negative Γ_0) and WA (positive Γ_0) regimes at finite length scales.** Length scale for the vortex dominated phase in the WA regime L_v (green line) and the disordered phase in the SA L_{BKT} (blue line) as a function of the anisotropy parameter, tunable by the drive, for the polariton system with $\delta_{CX} = -1.07$. The double red arrow A indicates a transition from the disordered WA to an algebraically ordered SA phase by increasing the external pump power in a system of size L such that $L_v < L < L_{BKT}$. The double red arrow B indicates a transition from the ordered WA regime to the ordered SA regime by increasing the external pump power in a system of size L , where $L < L_v < L_{BKT}$.

of meters for our *bad* cavity parameters. Thus scenario B is more likely in current experiments. Realising scenario A would require increasing the dissipation in a controlled way so that not all the collective effects are washed out and the OPO survives.

Finally, we can ask whether a transition between WA ($\Gamma > 0$) and SA ($\Gamma < 0$) regimes is possible in conditions of strong KPZ non-linearity ($g > 1$) where KPZ scaling would show at all lengthscales beyond the healing length. This would mean crossing a phase with stretched exponential decay of correlations to a phase with algebraic order as in equilibrium systems below the BKT transition. We find that such KPZ to algebraic-order crossover as a function of pump power is indeed possible at finite detuning (see Fig. 8 for the *bad* cavity system).

E. Different experimental systems

In the previous section we studied in detail the *bad* cavity configuration characterised by relatively large photon decay rate with a polariton lifetime of $\tau \approx 6.6$ ps, as the most promising for observation of different phases. However, current state-of-the-art inorganic microcavities are characterised by much longer polariton lifetimes. Indeed, *typical* inorganic cavities have polariton lifetimes of $\tau \approx 30$ ps [54], whereas the best cavities show polariton lifetimes $\tau \approx 150$ ps [52, 55]. We refer to the latter as

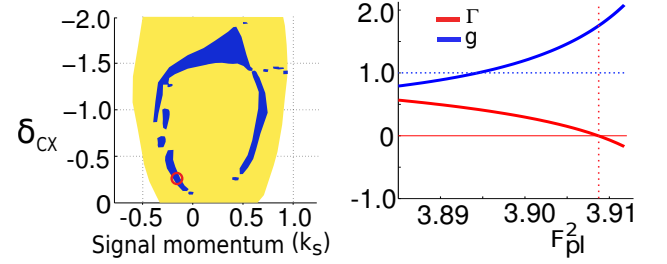


FIG. 8. **Crossover between the KPZ scaling and the SA (negative Γ_0) regime with algebraic order.** *Left panel:* Regions of detuning between the cavity photons and the excitons δ_{CX} and signal momentum k_s (dark blue) for which the system can exhibit a crossover between the KPZ phase and algebraically ordered phase by tuning the pump strength F_p . Yellow region marks stable KPZ-solutions. *Right panel:* g and Γ as a function of the normalized pump strength F_{pl}^2 for $k_s = -0.15$ and $\delta_{CX} = -0.25$ (small red circle in the left pannel). The crossover between the KPZ and the SA regimes occurs at $F_{pl}^2 \approx 3.909$: Γ changes sign (vertical red dotted line) while $g > 1$ (dotted blue horizontal line). $k_p = 1.4$ and we consider all values of k_s for which there is a non-zero mean field solution and a stable aKPZ equation. **Note:** Stability analysis indicates that in the blue regions the three-mode ansatz is unstable. This is confirmed by numerical studies [53], which show that the steady-state polariton field can develop several secondary modes in addition to the three main ones. However, the secondary modes are typically orders of magnitude weaker, and we expect them to give only minor quantitative corrections to the values of g and Γ we find here.

good cavities. Most inorganic samples are characterised by Rabi splittings Ω_R and exciton-exciton interaction-strengths g_X comparable to the ones used in the previous section for the *bad* cavity, i.e., $\hbar\Omega_R \approx 4.4$ meV and $g_X \approx 0.002$ meV μm^{-2} , respectively. On the other hand, *organic* microcavities have extremely low photon lifetime, but high Rabi splittings and relatively small exciton-exciton interaction strength. Typical values for organic cavities are $\tau \approx 5.5 \cdot 10^{-2}$ ps, $\hbar\Omega_R \approx 1000$ meV and $g_X \approx 10^{-6}$ meV μm^{-2} [56].

In this section we present a comparison of the relevant length scales for these different cavities, i.e., the *bad*, *typical*, *good* and organic cavities with zero detuning between the cavity-photons and the excitons. We first focus on the OPO region at high pump powers (region A in Fig. 2). The results are listed in Table I. The first four rows show: the length scale for the vortex dominated phase (L_v), the KPZ-phase (L_{KPZ}), the healing length (ξ_0), and the corresponding normalized pump power F_{np}^2 at a point where the L_v is smallest within stable three-mode OPO solutions (see left panel in Fig. 5 as an example for the *bad* cavity). The estimate for L_v is done using $\xi_0 \approx \bar{\xi}_0$ (cf. the discussion around Eq. (19), which may not be realistic, so this information has to be taken with a grain of salt. In addition, we observe an L_v which takes reasonable physical values for the *bad* cavity and organic samples, whereas the KPZ scale L_{KPZ} is unreachable.

As we mentioned in the last section, close to the upper threshold, the KPZ length scale drops down significantly, reaching ξ_0 and L_v (See right panel in Fig. 5 as an example for the *bad* cavity. Also, note that L_v is expected to be strongly renormalized close to the threshold as explained in Appendix B.). The 5th, 6th and 7th rows show the values of L_{KPZ} , ξ_0 and F_{lu}^2 at this point. We obtain that only for the *bad* and *typical* configurations, this intersection point can be distinguished from the upper threshold. The last row shows the magnitude of the drift term of the aKPZ equation in dimensionless units for $k_p = 1.4$. The organic cavity has the highest values, whereas the typical configuration the lowest one. We can conclude that in the state-of-the-art microcavities near the OPO threshold the length scales associated with the KPZ or vortex dominated phases are absolutely unrealistic and the physics is dominated by the equilibrium-like BKT transition between disorder and algebraically ordered phases.

Finally, we examine whether the strong KPZ non-linearity, which would result in stretched exponential decay of correlations at all lengthscales beyond the healing length (cf. Sec. III C) is present also in other cavity configurations. We find that for longer lifetime cavities this regime moves to higher pump powers above the OPO threshold with respect to the *bad* cavity configuration. However, for all cavities other than the *bad* cavity this regime falls into a region, where the three mode ansatz is unstable towards more complex solutions. Examining whether the KPZ scaling persists beyond the three mode ansatz is beyond the scope of this work.

F. Summary

We explored the wealth of scaling regimes accessible with coherently driven microcavity polaritons. The basis of our analysis is a long-wavelength effective description of OPO polaritons in terms of the compact anisotropic KPZ equation (9). A key point is that while the dynamics of both incoherently and coherently driven polaritons can be mapped to the aKPZ equation, in the latter case a much wider range of parameters is accessible by tuning the pump strength, the exciton-photon detuning, and the pump wave vector. Ultimately, the reason for this high versatility of OPO polaritons is different physics leading to formally the same long-wavelength description. In particular, the strongly fluctuating Goldstone mode is the phase of the condensate in the case of incoherent pumping, while it is the relative phase of signal and idler modes for polaritons in the OPO regime. The crucial merit of this high tunability is that the rich scaling behaviour of the compact anisotropic KPZ equation becomes accessible in a single experimental platform.

Experimentally, the scaling regimes can be distinguished by measuring the spatial and temporal decay of the first order coherence function (4): in the WA regime, strongly non-linear fluctuations of the phase lead

| Name | <i>bad</i> | Typical | Good | Organic |
|-------------------------------|-------------|-------------|-------------|----------------|
| $L_v(\mu\text{m})$ | 10^2 | 10^8 | 10^{27} | $2 \cdot 10^2$ |
| $L_{\text{KPZ}}(\mu\text{m})$ | 10^{10^3} | 10^{10^4} | 10^{10^5} | 10^{10^8} |
| $\xi_0(\mu\text{m})$ | 10 | 10 | 10 | 1 |
| F_{np}^2 | 0.9 | 0.84 | 0.97 | 0.84 |
| $L_{\text{KPZ}}(\mu\text{m})$ | 10^3 | 10^{10} | | |
| $\xi_0(\mu\text{m})$ | 60 | 10^2 | | |
| F_{pu}^2 | 0.9988 | 0.9999 | 1.0000 | 1.000 |
| $B_d(\mu\text{m}/\text{ps})$ | 0.30 | 0.18 | 0.30 | 2.69 |

TABLE I. **Comparison of different experimental microcavities.** Each column contains details for one of the four different considered cavities: *bad*, *typical*, *good* and *organic* samples for $k_p = 1.4$ and $k_s = 0.1$. The first four rows indicate characteristic length scales for the vortex dominated phase (L_v), for the KPZ-phase (L_{KPZ}), the healing length (ξ_0), and normalized pump power (F_{pu}^2) at the point where L_v reaches the minimum value in the stable three-mode OPO region close to the upper threshold (region A in Fig. 2). The 5th, 6th and 7th rows show L_{KPZ} , ξ_0 and F_{pu}^2 where $L_{\text{KPZ}} \approx L_v$, close to the upper threshold. The blank spaces for the good and the organic samples mean that this crossing point occurs practically at the upper threshold. The last row shows the values of the drift term B_d in dimensional units, which is approximately constant for the different values of the external drive (see Fig. 10 for the *bad* cavity case). It is clear that the KPZ lengthscale is unrealistic in regions A and C of the phase diagram presented in Fig. 2, leaving region B as the only potentially promising regime.

to stretched exponential decay of correlations, with exponents in the spatial and temporal “directions” given by the KPZ roughness and dynamical exponents. However, the non-linearity also leads to screening of the interactions of vortices which could result in their unbinding and thus preclude the observation of KPZ scaling as discussed in Appendix B. Either way, both effects are beyond the physics of 2D superfluids in equilibrium and it would be intriguing to see them in experiments. They occur beyond length and time scales that are exponentially large in the KPZ non-linearity. Thus, their observation is greatly facilitated in OPO polaritons, in which this non-linearity can become of order one. As the OPO threshold is approached in the WA regime, order on shorter scales is destroyed through the usual KT mechanism of vortex proliferation.

In the SA regime, the non-linearity is irrelevant, and the effective long-wavelength theory becomes linear as in thermal equilibrium. Thus, true algebraic order is possible.⁸ The effective renormalized noise level that determines whether the system is in the ordered or disordered phase depends in a non-trivial way on the pumping strength. In particular, it shows reentrant behaviour: upon increasing the driving strength the system can first

⁸ See footnote 7.

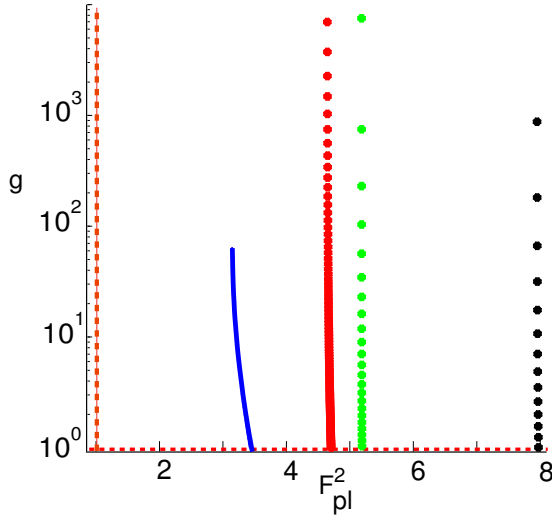


FIG. 9. **Regions potentially exhibiting KPZ phase for different experimental microcavities.** KPZ non-linearity parameter g (in a region where $g > 1$) as a function of the normalized pump strength F_{pl}^2 for four different microcavities, *bad* (blue line), *typical* (red dots), *good* (black dots) and organic (green dots) cavities. We present the regions which potentially could show KPZ scaling at all lengthscales beyond the healing length, i.e. $g > 1$ and $\Gamma > 1$ (see Sec. III A). The dashed horizontal line marks the $g = 1$ border, whereas the dashed vertical line indicates the normalized lower mean field threshold, i.e. $F_{pl}^2 = 1$. Note that only for the *bad* cavity the three-mode ansatz is stable in shown region. In this plot we consider $k_p = 1.4$ and $k_s = 0.1$ for the *bad* and *good* cavities and $k_p = 1.6$ and $k_s = 0.06$ for the *typical* and organic cavities.

enter the ordered phase and then leave it again.

The most intriguing prospect is thus to cross from the WA to the SA regime simply by tuning the pumping strength. Order is then established because of a change in the effective degree of anisotropy. This is shown to be in principle possible close to the upper OPO threshold for negative detuning (i.e. blue curve in Fig. 3). Then, as the pumping strength is increased, the system crosses from WA to SA, and within the SA regime from disordered to ordered and back. Another possibility to enter the SA regime close to the lower OPO threshold is to increase the pump momentum. This shows that all universal scaling regimes of the compact anisotropic KPZ equation are in principle accessible with OPO polaritons — in a sufficiently large system.

The relevant length scales above which asymptotic universal behaviour can be observed are determined by the strength of the non-linearity in the KPZ equation — a stronger non-linearity implies shorter crossover scales and is thus favourable for experimental observation. The mere presence of non-linearity in turn reflects that the system is fundamentally out of thermal equilibrium, and its value is enhanced if the dynamics is dominated by drive and dissipation, i.e., when polaritons have short lifetimes strong pumping is required. Our estimates for

the relevant scales to enter universal scaling regimes thus focus on a *bad* cavity (values of parameters are given at the beginning of Sec. III). We find that while close to the OPO threshold both L_{KPZ} and L_v , beyond which we expect KPZ scaling and unbinding of vortices (induced by the screening of interactions due to the KPZ non-linearity), respectively, are beyond realistic system sizes, there is a very promising regime away from the lower and upper thresholds in which non-linear effects are strong and L_{KPZ} and L_v are of the order of the condensate healing length. It would be desirable to explore this regime experimentally or by solving the stochastic equations of motion (7) numerically.

Finally, we discuss the crossover from WA to SA in finite-size systems, while approaching the upper threshold. The most interesting scenario would be to make a transition from a disordered to an algebraically ordered phase (up to the size of the system) by tuning the effective anisotropy from weak to strong (arrow A in Fig. 7). However, implementing this with the *bad* cavity parameters is not feasible in realistic system sizes. On the other hand, by optimising the exciton-photon detuning and moving to higher pump powers (around four times the lower OPO threshold in Fig. 6) we can cross from WA to SA regimes in conditions where the dimensionless KPZ non-linearity is larger than one implying KPZ scaling at all lengthscales beyond the healing length. From the experimental point of view, this means crossing from a KPZ phase with stretched exponential decay of spatial and temporal coherence to an algebraically ordered phase by increasing the pump strength in a controlled way.

IV. CONCLUSIONS

Exploring a fruitful example of parametrically driven microcavity polaritons, we have shown that the underlying order in highly driven and dissipative conditions heavily depends on the details of the driving process. In particular, a subtle interplay between dissipation and spatial inhomogeneity, controlled externally, allows one to move between different phases with different universal properties.

In our example, polaritons in the OPO regime, despite their intrinsic driven-dissipative nature and highly non-thermal occupations, can be driven to a phase, characterised by the equilibrium EW universality class, and thus become indistinguishable at asymptotic length scales from an equilibrium system, showing algebraic order and superfluidity even in the thermodynamic limit. This effect roots in the strong anisotropy that is feasible in the OPO configuration, and is in a stark contrast to, for example, incoherently driven polaritons, where algebraic order and superfluidity can only be a finite size effect, and will not survive in the thermodynamic limit. However, in the same OPO regime but at lower pump powers, the physics is governed by the non-equilibrium fixed point and KPZ universality class. Again, in con-

trast to incoherently pumped polaritons, in the middle region of the OPO phase diagram the KPZ length scales become small, up to the order of the healing length, suggesting that the KPZ order in a quantum system might indeed be observed in experiments on semiconductor microcavities. Our findings undoubtedly highlight the importance of the details of the driving mechanism in establishing the relevant order, applicable to a wide range of collective light-matter systems, and shine a new light on the ongoing debate about the nature of the polariton ordered phase in semiconductor microcavities.

V. ACKNOWLEDGEMENTS

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Appendix A: Mapping to the KPZ equation and numerical results

In this section, we consider in detail the mapping in the long-range limit between the stochastic equations (7), describing the dynamics of the OPO condensate, and the aKPZ equation (9). Firstly, as mentioned in Sec. II B, we introduce the hydrodynamic phase-amplitude representation (8) into the dynamical stochastic equations (7). In particular, considering the U(1) symmetry of the system (see Eq. (1)), it is convenient to change from the phase variables $\{\theta_s, \theta_i\}$ to $\{\theta_+, \theta\}$:

$$\begin{aligned}\theta_s &= \theta + \theta_+, & \theta &= \frac{1}{2}(\theta_s - \theta_i), \\ \theta_i &= -\theta + \theta_+, & \theta_+ &= \frac{1}{2}(\theta_s + \theta_i),\end{aligned}\quad (\text{A1})$$

where θ indicates the phase difference between the signal and the idler states. Thus, we get the following set of coupled dynamical equations for the fluctuations:

$$\begin{cases} \partial_t \theta + D_s(\theta) + \partial_t(\theta_+ - i\pi_s/\sqrt{\rho_s}) = M_s \phi - \tilde{\xi}_s, \\ \partial_t \theta + D_i(\theta) + \partial_t(-\theta_+ + i\pi_i/\sqrt{\rho_i}) = M_i \phi + \tilde{\xi}_i, \\ \partial_t(\theta_p - i\pi_p/\sqrt{\rho_p}) = M_p \phi - \tilde{\xi}_p, \end{cases} \quad (\text{A2})$$

where $\tilde{\xi}_j = \xi_j/(\sqrt{\rho_j}e^{i\varphi_j})$, and we have introduced the phase variables θ, θ_+ in place of θ_s, θ_i according to (A1).

The phase θ is kept to all orders whereas the fluctuations grouped together under $\phi^T = (\pi_s \ \pi_i \ \pi_p \ \theta_+ \ \theta_p)$ are considered up to linear order. M_j is the mass matrix of the mode j and its coefficients depend on the mean field parameters of the system. The operators $D_s(\theta) = -i\omega_{1s}\nabla\theta + i\nabla^T\omega_{2s}\nabla\theta + \nabla^T\theta\omega_{2s}\nabla$ and $D_i(\theta) = -i\omega_{1i}\nabla\theta + i\nabla^T\omega_{2i}\nabla\theta - \nabla^T\theta\omega_{2i}\nabla$ are obtained by considering the expansion of the polariton dispersion relation around \mathbf{k}_s and \mathbf{k}_i (see Eq. (10)).

Due to the U(1) symmetry of the system Eq. (1), the phase fluctuations θ is a gapless mode (there is no 'mass' term for such a fluctuation), whereas the fluctuations grouped under ϕ are all gapped. Note, that we neglect the following terms containing temporal and spatial derivatives: $\pi_j\partial_t\theta_j, \nabla\pi_j, \nabla^T\omega_{2j}\nabla\pi_j, \pi_j\nabla\theta_j, \pi_j\nabla^T\omega_{2j}\nabla\theta_j, \pi_j\nabla\theta_j$ and $\pi_j\nabla^T\theta_j\omega_{2j}\nabla\theta_j$, which are small compared to the mass-like contributions in the long range limit of π_j . Applying the same criterion, we also omit the spatial derivatives of θ_+ and θ_p . The next step is to consider the real and imaginary parts of Eq. (A2), and neglect the time derivatives of the massive modes, since they are 'slow' variables [16]:

$$\begin{pmatrix} \partial_t\theta + \text{Re}[D_s(\theta) + \tilde{\xi}_s] \\ \text{Im}[D_s(\theta) + \tilde{\xi}_s] \\ \partial_t\theta + \text{Re}[D_i(\theta) - \tilde{\xi}_i] \\ \text{Im}[D_i(\theta) - \tilde{\xi}_i] \\ \text{Re}[\tilde{\xi}_i] \\ \text{Im}[\tilde{\xi}_i] \end{pmatrix} = \begin{pmatrix} \text{Re}[M_s] \\ \text{Im}[M_s] \\ \text{Re}[M_i] \\ \text{Im}[M_i] \\ \text{Re}[M_p] \\ \text{Im}[M_p] \end{pmatrix} \phi. \quad (\text{A3})$$

In (A3) the massive fluctuations can be eliminated, i.e. they can be expressed solely in terms of the spatial and time derivatives of θ , and the noise terms. By considering the last five relations of (A3), we can express the gapped modes as:

$$\phi^T = \phi^T(\partial_t\theta, \partial_x\theta, \partial_y\theta, \partial_x^2\theta, \partial_y^2\theta, \tilde{\xi}_s, \tilde{\xi}_i, \tilde{\xi}_p), \quad (\text{A4})$$

and substitute into the first equation of (A3) to obtain a single dynamical stochastic equation for the gapless θ variable, which has a form of the KPZ equation with a drift term (Eq. (9)). The different KPZ coefficients in (9) read:

$$\begin{aligned} D_m &= \alpha_1\omega_{2s,mm} + \alpha_2\omega_{2i,mm}, \\ \lambda_m &= \alpha_3\omega_{2s,mm} + \alpha_4\omega_{2i,mm}, \\ B_m &= \alpha_5\omega_{1s,m} + \alpha_6\omega_{1i,m}, \\ 2\Delta &= (\beta_1^2 + \beta_2^2)\gamma_s + (\beta_3^2 + \beta_4^2)\gamma_i + (\beta_5^2 + \beta_6^2)\gamma_p \end{aligned} \quad (\text{A5})$$

for $m = x, y$. The α and β coefficients come from (A4) while the ω coefficients originate from the lower polariton dispersion (10): $\omega_{1j}^T = (\omega_{1j,x}, \omega_{1j,y})$ and

$$\omega_{2j} = \begin{pmatrix} \omega_{2j,xx} & 0 \\ 0 & \omega_{2j,yy} \end{pmatrix}. \quad (\text{A6})$$

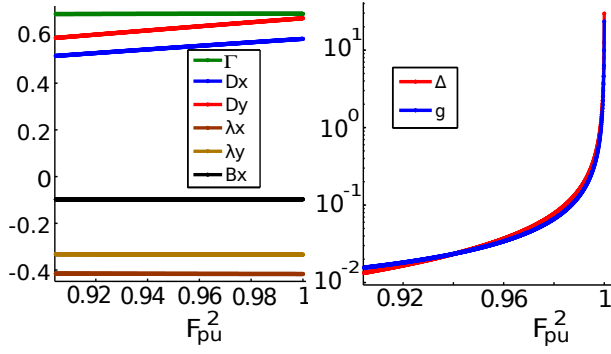


FIG. 10. **aKPZ coefficients for the bad cavity configuration with zero-detuning.** *Left panel:* Diffusion coefficients (D_x, D_y), non-linearities (λ_x, λ_y), anisotropy parameter (Γ), and drift term B_x as a function of the pump power normalised to the upper threshold. These parameters vary little with the external pump power. *Right panel:* Dimensionless non-linearity g and noise parameter Δ as a function of the normalised pump power. We observe the asymptotic growth of these parameters close to the upper OPO threshold.

As an example, in Fig. 10, we show the numerical values of the different KPZ coefficients as a function of pump power close to the OPO upper threshold and at zero detuning for the set of parameters given in Sec. III characterising what we call a *bad* cavity. The system is WA in the region shown since λ s have the same sign. The drift term acts in the x-direction, due to the choice of the pumping wave-vector. i.e. $\mathbf{k}_p = (k_p, 0)$. In this WA regime, the KPZ coefficients do not vary excessively with respect to the external pump power apart from g and Δ which asymptote to infinity at the upper mean-field OPO threshold.

Appendix B: Vortices in the compact KPZ equation in the WA regime

In thermal equilibrium, vortices of opposite charge attract each other with a force that falls off as $\sim 1/r$, like charges in a 2D Coulomb gas. This leads to the formation of closely bound dipoles at low temperatures, whereas at high temperatures the interaction is screened at large distances and the bound state of vortex-antivortex pairs is no longer stable. The fundamental qualitative modification in a driven-dissipative system is that due to the KPZ non-linearity the vortex interaction is screened even in the absence of noise-induced fluctuations [17] (a situation corresponding to zero temperature in an equilibrium problem). Therefore, even without noise there is a finite screening length beyond which the interaction is suppressed exponentially. For weak non-linearities, this length scale is given by expression (19).

As explained above, the screening of the interaction between vortices beyond the scale L_v is solely due to the non-linear terms in the KPZ equation. At finite noise, when fluctuations lead to the creation of vortex-

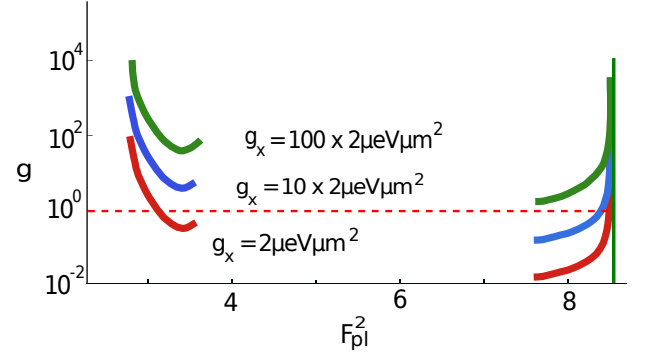


FIG. 11. **Dependence of the non-linear parameter g on exciton-exciton interaction strengths.** g for three different values of g_x as a function of the normalized external pump power F_{pl} with the lower threshold for a cavity with zero detuning. We observe that the systems shows larger values of g when increasing the exciton-exciton interaction.

antivortex pairs, there is additional screening induced by the polarization of bound pairs (which leads to unbinding above the critical temperature in the usual equilibrium BKT problem). This effect can only be captured in a proper RG treatment [19] and is not incorporated in the estimate Eq. (19). Associated with L_v is a time scale t_v for vortices to escape the region of attractive interactions at distances below L_v . Then, a vortex-dominated regime characterized by exponential decay of correlations and in which superfluidity is destroyed should appear above the scales L_v and t_v . If these scales are smaller than the corresponding KPZ scales defined above, then the scaling forms (16) and (17) will be completely masked by the vortex-induced exponential decay. Indeed, for weak KPZ non-linearities, some of us estimated that $L_v \ll L_{KPZ}$ and $t_v \ll t_{KPZ}$ [19]. The latter estimate for the time scales relies on the assumption that the mobility of vortices is not atypically small, i.e., not much smaller than the diffusion coefficients $D_{x,y}$ in the aKPZ equation (9), which are determined by the same microscopic physics.

Vortex unbinding induced by non-equilibrium conditions has so far remained elusive in experiments with incoherently pumped, and thus isotropic, polariton systems, as well as in the stochastic simulations described in [21]. This could be ascribed to the limited length and time scales available to experiments and numerics, but it could also be taken as an indication that the time scale t_v for vortices to unbind is indeed much larger than expected, thus leaving open the intriguing possibility to observe KPZ scaling if the system is initialized in a vortex-free state and parameters are chosen such that the dimensionless non-linearity g is large (leading to small values of t_{KPZ} and L_{KPZ} , see Eq. (18)). The most promising regime for observing KPZ physics is described in Sec. III C.

Appendix C: Effects of the exciton-exciton coupling strength

Throughout this paper we considered the exciton-exciton interaction strength to be $g_X = 2 \mu\text{eV } \mu\text{m}^{-2}$. However, the true value of g_X is still subject of debates, and different values have been reported in literature (see for example [57]). We consider $g_X = 2 \mu\text{eV } \mu\text{m}^{-2}$ to be the lower bound, and the safe upper bound being 100 times this lower value. Thus, in this section we study the effect of larger values of the exciton-exciton interactions in two different configurations for the polariton system, characterized by detunings $\delta_{CX} = 0$ and $\delta_{CX} = -1.08$ in dimensionless units. We consider

three different values for the exciton-exciton interaction: $g_X = 2 \mu\text{eV } \mu\text{m}^{-2}$, $10 \cdot 2 \mu\text{eV } \mu\text{m}^{-2}$, $100 \cdot 2 \mu\text{eV } \mu\text{m}^{-2}$. In Fig. 11 we show the nonlinear parameter g as a function of the normalized external pump power (for zero detuning) and as a function of the anisotropy parameter Γ (for finite detuning). We observe that, by increasing the exciton-exciton interaction constant, the system is characterised by larger values of the non-linear parameter g for the same values of the external pump power (see left panel in Fig. 11). A similar phenomenon appears for finite detuning, where the value of g also increases by increasing g_X (see right panel in Fig. 11). The difference is, however, not large enough to alter the conclusions presented in the main text.

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